Differential Geometry
BMath-III Mid-Sem.test

Instructions : Attempt any five problems, they all carry equal weight. You may quote and use any result proved in the class.

1. Let $\gamma(t)$ be a unit speed curve in $\mathbb{R}^3$ with curvature $\kappa(t)$, nonzero for all t. Let the curve $\phi(t)$ be defined by

$$\phi(t) = \frac{d\gamma(t)}{dt}.$$

Prove that ϕ is a regular curve and the arc-length function $s(t)$ satisfies

$$\frac{ds}{dt} = \kappa.$$

2. Let γ be a unit speed plane curve with signed curvature κ_s, unit tangent vector $\mathbf{t}$ and signed unit normal $\mathbf{n_s}$. Show that $\frac{d\mathbf{n_s}}{dt} = -\kappa_s \mathbf{t}$.

3. Compute the area of the interior of an ellipse whose major and minor axes lengths have product equal to 1.

4. Calculate the distance between the points on the unit sphere given in their latitude-longitude coordinates (θ, ϕ) by $\mathbf{P} = (\theta_0, \phi_0) = (0, \pi/4))$ and $\mathbf{Q} = (\theta_1, \phi_1) = (\pi/4, \pi/4))$. Show by an example that the distance between two points on a surface in $\mathbb{R}^3$ is, in general, different from the Euclidean distance between them.

5. Show by an example that conformal maps between surface patches are not isometries in general.

6. Let $\gamma(u)$ be a simple closed curve with period a, unit tangent $\mathbf{t}$, signed unit normal $\mathbf{n_s}$ and signed curvature κ_s. Show that

$$\mathbf{t}(u+a) = \mathbf{t}(u), \quad \mathbf{n_s}(u+a) = \mathbf{n_s}(u), \quad \kappa_s(u+a) = \kappa_s(u).$$

(Hint : differentiate the equation $\gamma(u+a) = \gamma(u)$)